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The presentation of a new type of quantum calculus

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Abstract

In this paper we introduce a new type of quantum calculus, the p-calculus involving two concepts of p-derivative and p-integral. After familiarity with them some results are given.

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1 Introduction

Simply put, quantum calculus is ordinary calculus without taking limit. In ordinary calculus, the derivative of a function $f(x)$ is defined as $f'(x) = \lim_{y\to x} \frac{f(y)-f(x)}{y-x}$. However, if we avoid taking the limit and also take $y = x^p$, where p is a fixed number different from 1, i.e., by considering the following expression:

$$\frac{f(x^p) - f(x)}{x^p - x}, \tag{1.1}$$

then, we create a new type of quantum calculus, the p-calculus, and the corresponding express is the definition of the p-derivative. The formula (1.1) and several of the results derived from it which will be mentioned in the next sections, appear to be new. In [8] the authors developed two types of quantum calculus, the q-calculus and the h-calculus. If in the definition of $f'(x)$, as has been stated above, we do not take limit and also take $y = qx$ or $y = x + h$, where q is a fixed number different from 1, and h a fixed number different from 0, the q-derivative and the h-derivative of $f(x)$ are defined. For more details, we refer the readers to [1, 2, 4, 7]. Generally, in the last decades the q-calculus has developed into an interdisciplinary subject, which is briefly discussed in chapters 3 and 7 of [3] and also has interesting applications in various sciences such as physics, chemistry, etc [5, 6]. A history of the q-calculus was given by T.Ernst [3].

The purpose of this paper is to introduce another type of quantum calculus, the p-calculus, also we're going to give some results by it. The paper has been organized as follows. In section 2, we define the p-derivative, also some of its properties will be expressed. In section 3, we introduce the p-integral, including a sufficient condition for its convergence is given. In section 4, we will define the definite p-integral, followed by the definition of the improper p-integral. Finally, we will conclude our discussion by fundamental theorem of p-calculus.

2 p-Derivative

Throughout this section, we assume that p is a fixed number different from 1 and domain of function $f(x)$ is $[0, +\infty)$.

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Definition 2.1. Let $f(x)$ be an arbitrary function. We define its p-differential to be

$$d_p f(x) = f(x^p) - f(x).$$

In particular, $d_p x = x^p - x$. By the p-differential we can define p-derivative of a function.

Definition 2.2. Let $f(x)$ be an arbitrary function. We define its p-derivative to be

$$D_p f(x) = \frac{f(x^p) - f(x)}{x^p - x}, \qquad if x \neq 0, 1$$

and

$$D_p f(0) = \lim_{x \to 0^+} D_p f(x), \qquad D_p f(1) = \lim_{x \to 1} D_p f(x).$$

Remark 2.3. If $f(x)$ is differentiable, then $\lim_{p \to 1} D_p f(x) = f'(x)$, and also if $f'(x)$ exists in a neighborhood of $x = 0$, $x = 1$ and is continuous at $x = 0$ and $x = 1$, then we have

$$D_p f(0) = f'_+(0), \qquad D_p f(1) = f'(1).$$

Definition 2.4. The p-derivative of higher order of function f is defined by

$$(D_p^0 f)(x) = f(x), \qquad (D_p^n f)(x) = D_p(D_p^{n-1} f)(x), n \in N.$$

Example 2.5. Let $f(x) = c$, $g(x) = x^n$ and $h(x) = \ln(x)$ where c is constant and $n \in N$. Then we have

(i) $D_p f(x) = 0$,

(ii) $D_p g(x) = \dfrac{g(x^p) - g(x)}{x^p - x} = \dfrac{x^{pn} - x^n}{x^p - x} = \dfrac{x^{(p-1)n} - 1}{x^{p-1} - 1} x^{n-1}$,

(iii) $D_p h(x) = \dfrac{h(x^p) - h(x)}{x^p - x} = \dfrac{(p-1)\ln(x)}{x^p - x} = \dfrac{(p-1)\ln(x)}{x^{p-1} - 1} \dfrac{1}{x}$.

Notice that the p-derivative is a linear operator, i.e., for any constants a and b, and arbitrary functions $f(x)$ and $g(x)$, we have

$$D_p(af(x) + bg(x)) = aD_p f(x) + bD_p g(x).$$

We want now to compute the p-derivative of the product and the quotient of $f(x)$ and $g(x)$.

$$\begin{aligned} D_p(f(x)g(x)) &= \frac{f(x^p)g(x^p) - f(x)g(x)}{x^p - x} \\ &= \frac{f(x^p)g(x^p) - f(x)g(x^p) + f(x)g(x^p) - f(x)g(x)}{x^p - x} \\ &= \frac{(f(x^p) - f(x))g(x^p) + f(x)(g(x^p) - g(x))}{x^p - x}. \end{aligned}$$

Thus

$$D_p(f(x)g(x)) = g(x^p)D_p f(x) + f(x)D_p g(x). \tag{2.1}$$

Similarly, we can interchange f and g, and obtain

$$D_p(f(x)g(x)) = g(x)D_pf(x) + f(x^p)D_pg(x), \tag{2.2}$$

which both of (2.1) and (2.2) are valid and equivalent. Here let us prove quotient rule. By changing $f(x)$ to $\frac{f(x)}{g(x)}$ in (2.1), we have

$$D_pf(x) = D_p(\frac{f(x)}{g(x)}g(x)) = g(x^p)D_p(\frac{f(x)}{g(x)}) + \frac{f(x)}{g(x)}D_pg(x),$$

and thus

$$D_p(\frac{f(x)}{g(x)}) = \frac{g(x)D_pf(x) - f(x)D_pg(x)}{g(x)g(x^p)}. \tag{2.3}$$

Using (2.2) with functions $\frac{f(x)}{g(x)}$ and $g(x)$, we obtain

$$D_p(\frac{f(x)}{g(x)}) = \frac{g(x^p)D_pf(x) - f(x^p)D_pg(x)}{g(x)g(x^p)}. \tag{2.4}$$

Both of (2.3) and (2.4) are valid.

Note 2.6. We do not have a general chain rule for p-derivatives, but in most cases we can have the following rule:

$$D_p[f(u(x))] = D_pu(x)D_{\frac{h(x)}{\ln(u(x))}}f(u(x)),$$

where $h(x)$ is depended on $u(x)$.

Example 2.7. If $\alpha > 0$ and $u(x) = \alpha x^\beta$, then

$$\begin{aligned} D_p[f(u(x))] &= \frac{f(\alpha x^{p\beta}) - f(\alpha x^\beta)}{x^p - x} \\ &= \frac{f(\alpha x^{p\beta}) - f(\alpha x^\beta)}{\alpha x^{p\beta} - \alpha x^\beta} \cdot \frac{\alpha x^{p\beta} - \alpha x^\beta}{x^p - x} \\ &= D_{\frac{\ln(\alpha)+p\beta\ln(x)}{\ln(u(x))}}f(u(x))D_pu(x), \end{aligned}$$

because, $u(x)^{\frac{\ln(\alpha)+p\beta\ln(x)}{\ln(u(x))}} = \alpha x^{p\beta}$.

Example 2.8. If $\alpha > 0$ and $u(x) = \alpha e^x$, then

$$\begin{aligned} D_p[f(u(x))] &= \frac{f(\alpha e^{x^p}) - f(\alpha e^x)}{x^p - x} \\ &= \frac{f(\alpha e^{x^p}) - f(\alpha e^x)}{\alpha e^{x^p} - \alpha e^x} \cdot \frac{\alpha e^{x^p} - \alpha e^x}{x^p - x} \\ &= D_{\frac{\ln(\alpha)+x^p}{\ln(u(x))}}f(u(x))D_pu(x), \end{aligned}$$

because, $u(x)^{\frac{\ln(\alpha)+x^p}{\ln(u(x))}} = \alpha e^{x^p}$.

3 p-Integral

The first thing that comes to our mind after studying the derivative of a function is its integral topic. Before investigating it, let us define p-antiderivative of a function.

Definition 3.1. A function $F(x)$ is a p-antiderivative of $f(x)$ if $D_pF(x) = f(x)$. It is denoted by

$$F(x) = \int f(x)d_px.$$

Notice that as in ordinary calculus, the p-antiderivative of a function might not be unique. We can prove the uniqueness by some restrictions on the p-antiderivative and on p.

Theorem 3.2. Suppose $0 < p < 1$. Then, up to adding a constant, any function $f(x)$ has at most one p-antiderivative that is continuous at $x = 1$.

Proof. Suppose F_1 and F_2 are two p-antiderivative of $f(x)$ that are continuous at $x = 1$. Let $\Phi(x) = F_1(x) - F_2(x)$. Since F_1 and F_2 are continuous at $x = 1$ and also by the definition of p-derivative that lead to $D_p\Phi(x) = 0$, we have Φ is continuous at $x = 1$ and $\Phi(x^p) = \Phi(x)$ for any x. Since for some sufficiently large $N > 0$, $\Phi(x^{p^N}) = \Phi(x^{p^{N+1}}) = ... = \Phi(x)$ and also by the continuity Φ at $x = 1$, it follows that $\Phi(x) = \Phi(1)$. □

As was mentioned we denote the p-antiderivative of $f(x)$ by function $F(x)$ such that $D_pF(x) = f(x)$. Here we're going to construct the p-antiderivative. For this purpose, we use of an operator. We define an operator $\hat{M}_p$, by $\hat{M}_p(F(x)) = F(x^p)$. Then we have:

$$\frac{1}{x^p - x}(\hat{M}_p - 1)F(x) = \frac{F(x^p) - F(x)}{x^p - x} = D_pF(x) = f(x).$$

Since $\hat{M}_p^j(F(x)) = F(x^{p^j})$ for $j \in \{0, 1, 2, 3, ...\}$ and also by the geometric series expansion, we formally have

$$F(x) = \frac{1}{1 - \hat{M}_p}((x - x^p)f(x)) = \sum_{j=0}^{\infty} \hat{M}_p^j((x - x^p)f(x)) = \sum_{j=0}^{\infty}(x^{p^j} - x^{p^{j+1}})f(x^{p^j}). \tag{3.1}$$

It is worth mentioning that we say that (3.1) is formal because the series does not always converge.

Definition 3.3. The p-integral of $f(x)$ is defined to be the series

$$\sum_{j=0}^{\infty}(x^{p^j} - x^{p^{j+1}})f(x^{p^j}). \tag{3.2}$$

Remark 3.4. Generally, the p-integral does not always converge to a p-antiderivative. Here we want to give a sufficient condition for convergence the p-integral to a p-antiderivative.

Theorem 3.5. Suppose $0 < p < 1$. If $|f(x)x^{\alpha}|$ is bounded on the interval $(0, A]$ for some $0 \leq \alpha < 1$, then the p-integral defined by (3.2) converges to a function $H(x)$ on $(0, A]$, which is a p-antiderivative of $f(x)$. Moreover, $H(x)$ is continuous at $x = 1$ with $H(1) = 0$.

Proof. We consider the following two cases.

Case 1. $x \in (1, A]$. Suppose $| f(x)x^{\alpha} |< M$ on $(1, A]$. For any $1 < x \leq A$, $j \geq 0$

$$| f(x^{p^j}) |< M(x^{p^j})^{-\alpha} < M.$$

Thus, for any $1 < x \leq A$, we have

$$| (x^{p^j} - x^{p^{j+1}})f(x^{p^j}) |\leq (x^{p^j} - x^{p^{j+1}})M.$$

Since

$$\sum_{j=0}^{\infty}(x^{p^j} - x^{p^{j+1}})M = M(x-1),$$

thus, it follows from the comparison test that the p-integral converges to a function $F(x)$. It follows directly from (3.1) that $F(1) = 0$. We want now to prove that $F(x)$ is a p-antiderivative of $f(x)$, but before of it let us show F is right continuous at $x = 1$. For $1 < x \leq A$,

$$| F(x) |=| \sum_{j=0}^{\infty}(x^{p^j} - x^{p^{j+1}})f(x^{p^j}) |\leq M(x-1),$$

which approaches 0 as $x \to 1^+$. Since $F(1) = 0$, thus F is right continuous at $x = 1$. To prove that $F(x)$ is a p-antiderivative, it is sufficient to p-differentiate it:

$$\begin{aligned} D_pF(x) &= \frac{F(x^p) - F(x)}{x^p - x} \\ &= \frac{\sum_{j=0}^{\infty}(x^{p^{j+1}} - x^{p^{j+2}})f(x^{p^{j+1}}) - \sum_{j=0}^{\infty}(x^{p^j} - x^{p^{j+1}})f(x^{p^j})}{x^p - x} \\ &= f(x). \end{aligned}$$

Case 2. $x \in (0, 1)$. Suppose $| f(x)x^{\alpha} |< M$ on $(0, 1)$. For any $0 < x < 1$, $j \geq 0$

$$| f(x^{p^j}) |< M(x^{p^j})^{-\alpha} \leq Mx^{-\alpha}.$$

Thus, for any $0 < x < 1$, we have

$$| (x^{p^j} - x^{p^{j+1}})f(x^{p^j}) |\leq (x^{p^{j+1}} - x^{p^j})Mx^{-\alpha},$$

and since

$$\sum_{j=0}^{\infty}(x^{p^{j+1}} - x^{p^j})Mx^{-\alpha} = Mx^{-\alpha}(1-x),$$

hence, it follows from the comparison test that the p-integral converges to a function $G(x)$ and by (3.1) we have $G(1) = 0$. Similar to proof of case 1, it is easy to verify that G is left continuous at $x = 1$ and is also a p-antiderivative of $f(x)$. We now define

$$H(x) = G(x)\chi_{(0,1)}(x) + F(x)\chi_{(1,A]}(x).$$

It is easy to see p-integral converges to $H(x)$ on $(0, A]$ and also $H(x)$ is a p-antiderivative of $f(x)$ on $(0, 1) \cup (1, A]$ and is continuous at $x = 1$ with $H(1) = 0$. If $f(x)$ is continuous in $x = 1$, then $D_pH(1) = \lim_{x\to 1} D_pH(x) = f(1)$ and it concludes that $H(x)$ is a p-antiderivative of $f(x)$ on $(0, A]$, hence the proof is complete. □

Corollary 3.6. If the assumption of Theorem 3.5 is satisfied, then by Theorem 3.2, the p-integral gives the unique p-antiderivative that is continuous at $x = 1$, up to adding a constant.

Example 3.7. Let $0 < p < 1$ and $f(x) = c$, i.e., $f(x)$ is constant. Since for $0 \le \alpha < 1$, $| f(x)x^\alpha |$ is bounded on interval $(0, A]$, hence by Theorem 3.5, p-integral of $f(x)$ converges whose it is valid, because

$$\sum_{j=0}^{\infty}(x^{p^j} - x^{p^{j+1}})f(x^{p^j}) = c\sum_{j=0}^{\infty}(x^{p^j} - x^{p^{j+1}}) = c(x-1)\chi_{(0,A]}(x).$$

Example 3.8. Let $0 < p < 1$ and $f(x) = \frac{1}{x - x^p}$. The p-integral gives

$$\sum_{j=0}^{\infty}(x^{p^j} - x^{p^{j+1}})f(x^{p^j}) = \sum_{j=0}^{\infty}(x^{p^j} - x^{p^{j+1}})\frac{1}{x^{p^j} - x^{p^{j+1}}} = \infty,$$

and also $f(x)x^\alpha$ is not bounded on $(0, 1) \cup (1, A]$ and $0 \le \alpha < 1$.

4 Definite p-Integral

We now are in position to define the definite p-integral. Generally, one of the principle tools to define the definite p-integral of a function is use of a partition on a set. we will use of it to achieve our goal. As proof of the Theorem 3.5, we consider the following three cases. Then, the definite p-integral related to each case is given.

Case 1. Let $1 < a < b$ where $a, b \in R^+$, $p \in (0, 1)$ and function f is defined on $(1, b]$. Notice that for any $j \in \{0, 1, 2, 3, ...\}$, $b^{p^j} \in (1, b]$. We now define the definite p-integral of $f(x)$ on interval $(1, b]$.

Definition 4.1. The definite p-integral of $f(x)$ on the interval $(1, b]$ is defined as

$$\int_1^b f(x)d_px = \lim_{N\to\infty}\sum_{j=0}^{N}(b^{p^j} - b^{p^{j+1}})f(b^{p^j}) = \sum_{j=0}^{\infty}(b^{p^j} - b^{p^{j+1}})f(b^{p^j}), \tag{4.1}$$

and

$$\int_a^b f(x)d_px = \int_1^b f(x)d_px - \int_1^a f(x)d_px. \tag{4.2}$$

Note 4.2. Geometrically, the integral in (4.1) corresponds to the area of the union of an infinite number of rectangles. On $[1+\varepsilon, b]$, where ε is a small positive number, the sum consists of finitely many terms, and is a Riemann sum. Therefore, as $p \to 1$, the norm of partition approaches zero, and the sum tends to the Riemann integral on $[1+\varepsilon, b]$. Since ε is arbitrary, provided that $f(x)$ is continuous in the interval $[1, b]$, thus we have

$$\lim_{p\to 1}\int_1^b f(x)d_px = \int_1^b f(x)dx.$$

Example 4.3. Let $b = 3$ and $f(x) = c$ where c is constant.

$$\begin{aligned}\int_1^3 cd_px &= \lim_{N\to\infty}\sum_{j=0}^{N}(3^{p^j} - 3^{p^{j+1}})f(3^{p^j})\\ &= c\lim_{N\to\infty}[(3-3^p) + (3^p - 3^{p^2}) + (3^{p^2} - 3^{p^3}) + ... + (3^{p^N} - 3^{p^{N+1}})]\\ &= c\lim_{N\to\infty}[3 - 3^{p^{N+1}}] = c(3-1) = 2c,\end{aligned}$$

and if $a = 2$,

$$\int_2^3 cd_px = \int_1^3 cd_px - \int_1^2 cd_px = 2c - c = c.$$

Example 4.4. Let $b = 2$ and $f(x) = \dfrac{\ln(x)}{x - x^p}$.

$$\int_1^2 f(x)d_px = \sum_{j=0}^{\infty}(2^{p^j} - 2^{p^{j+1}})\frac{\ln(2^{p^j})}{2^{p^j} - 2^{p^{j+1}}} = \sum_{j=0}^{\infty}p^j\ln(2) = \frac{\ln(2)}{1-p}.$$

Case 2. Let $0 < a < b < 1$ and $p \in (0, 1)$. It should be noted that for any $j \in \{0, 1, 2, 3, ...\}$, $b^{p^j} \in [b, 1)$ and $b^{p^j} < b^{p^{j+1}}$. We will define the definite p-integral of $f(x)$ on interval $[b, 1)$ as follows.

Definition 4.5. The definite p-integral of $f(x)$ on the interval $[b, 1)$ is defined as

$$\int_b^1 f(x)d_px = \lim_{N\to\infty}\sum_{j=0}^{N}(b^{p^{j+1}} - b^{p^j})f(b^{p^j}) = \sum_{j=0}^{\infty}(b^{p^{j+1}} - b^{p^j})f(b^{p^j}).$$

Example 4.6. Let $b = \frac{1}{2}$ and $f(x) = c$

$$\begin{aligned}\int_{\frac{1}{2}}^1 cd_px &= \lim_{N\to\infty}\sum_{j=0}^{N}((\frac{1}{2})^{p^{j+1}} - (\frac{1}{2})^{p^j})c\\ &= c\lim_{N\to\infty}[((\frac{1}{2})^p - (\frac{1}{2})) + ((\frac{1}{2})^{p^2} - (\frac{1}{2})^p) + ((\frac{1}{2})^{p^3} - (\frac{1}{2})^{p^2}) + ... + ((\frac{1}{2})^{p^{N+1}} - (\frac{1}{2})^{p^N})]\\ &= c\lim_{N\to\infty}[-\frac{1}{2} + (\frac{1}{2})^{p^{N+1}}] = c(-\frac{1}{2} + 1) = \frac{1}{2}c.\end{aligned}$$

Note 4.7. The above two definite p-integrals are also denoted by

$$\int_1^b f(x)d_px = I_{p^+}f(b),$$
$$\int_b^1 f(x)d_px = I_{p^-}f(b).$$

Case 3. Let $0 < a < b < 1$ and $p \in (0,1)$. Then for any $j \in \{0,1,2,3,...\}$, $b^{p^{-j}} \in (0,b]$ and $b^{p^{-j-1}} < b^{p^{-j}}$. Let us state the definite p-integral of $f(x)$ on interval $(0,b]$.

Definition 4.8. The definite p-integral of $f(x)$ on the interval $(0,b]$ is defined as

$$I_pf(b) = \int_0^b f(x)d_px = \lim_{N\to\infty}\sum_{j=0}^{N}(b^{p^{-j}} - b^{p^{-j-1}})f(b^{p^{-j-1}}) = \sum_{j=0}^{\infty}(b^{p^{-j}} - b^{p^{-j-1}})f(b^{p^{-j-1}}), \quad (4.3)$$

and

$$\int_a^b f(x)d_px = \int_0^b f(x)d_px - \int_0^a f(x)d_px. \quad (4.4)$$

Example 4.9. Let $a = \frac{1}{4}$, $b = \frac{1}{2}$ and $f(x) = c$.

$$\begin{aligned}\int_0^{\frac{1}{2}} cd_px &= \lim_{N\to\infty}\sum_{j=0}^{N}((\frac{1}{2})^{p^{-j}} - (\frac{1}{2})^{p^{-j-1}})c \\ &= c\lim_{N\to\infty}[((\frac{1}{2}) - (\frac{1}{2})^{p^{-1}}) + ((\frac{1}{2})^{p^{-1}} - (\frac{1}{2})^{p^{-2}}) + ... + ((\frac{1}{2})^{p^{-N}} - (\frac{1}{2})^{p^{-N-1}})] \\ &= c\lim_{N\to\infty}[(\frac{1}{2}) - (\frac{1}{2})^{p^{-N-1}}] = \frac{1}{2}c.\end{aligned}$$

Similarly,

$$\int_0^{\frac{1}{4}} cd_px = \frac{1}{4}c,$$

thus we have

$$\int_{\frac{1}{4}}^{\frac{1}{2}} cd_px = \int_0^{\frac{1}{2}} cd_px - \int_0^{\frac{1}{4}} cd_px = \frac{1}{4}c.$$

Note 4.10. We can also apply Note 4.2 for the p-integrals defined in the cases 2 and 3 on the intervals $[b, 1-\varepsilon]$ and $[\varepsilon, b]$ respectively, and by it define the Riemann integral.

Definition 4.11. Suppose $0 \le a < 1 < b$. Then by Note 4.2 and Note 4.10, we have

$$\int_a^b f(x)d_px = \int_a^1 f(x)d_px + \int_1^b f(x)d_px.$$

Corollary 4.12. By the definitions of p-integrals, we derive a more general formula:

i) If $b > 1$,

$$\int_1^b f(x) d_p g(x) = \sum_{j=0}^{\infty} f(b^{p^j})(g(b^{p^j}) - g(b^{p^{j+1}})).$$

ii) If $0 < b < 1$,

$$\int_0^b f(x) d_p g(x) = \sum_{j=0}^{\infty} f(b^{p^{-j-1}})(g(b^{p^{-j}}) - g(b^{p^{-j-1}})).$$

Because,

$$\begin{aligned}\int_1^b f(x) D_p g(x) d_p x &= \sum_{j=0}^{\infty} (b^{p^j} - b^{p^{j+1}})(f(b^{p^j}) D_p g(b^{p^j})) \\ &= \sum_{j=0}^{\infty} (b^{p^j} - b^{p^{j+1}}) f(b^{p^j})(\frac{g(b^{p^{j+1}}) - g(b^{p^j})}{b^{p^{j+1}} - b^{p^j}}) \\ &= \sum_{j=0}^{\infty} f(b^{p^j})(g(b^{p^j}) - g(b^{p^{j+1}})).\end{aligned}$$

Since $D_p g(x) = \frac{d_p g(x)}{d_p x}$, hence we have

$$\int_1^b f(x) d_p g(x) = \sum_{j=0}^{\infty} f(b^{p^j})(g(b^{p^j}) - g(b^{p^{j+1}})).$$

Similarly, it is easy to prove (b).

Definition 4.13. The p-integral of higher order of function f is given by

$$(I_p^0 f)(x) = f(x), \quad (I_p^n f)(x) = I_p(I_p^{n-1} f)(x), \;\; n \in N.$$

5 Improper p-Integral

In this section we want to define the improper p-integral of $f(x)$ and also give a sufficient condition for its convergence. We start this section by computing the following p-integral.

Let $p \in (0,1)$, thus $p^{-1} > 1$ and consider $p^{-1} = b$. For any $j \in \{0, \pm 1, \pm 2, ...\}$, we have $b^{p^j} > 1$, $b^{p^{j+1}} < b^{p^j}$ and thus according to (4.2), we obtain

$$\begin{aligned}\int_{b^{p^{j+1}}}^{b^{p^j}} f(x) d_p x &= \int_1^{b^{p^j}} f(x) d_p x - \int_1^{b^{p^{j+1}}} f(x) d_p x \\ &= \sum_{k=0}^{\infty} ((b^{p^j})^{p^k} - (b^{p^j})^{p^{k+1}}) f((b^{p^j})^{p^k}) - \sum_{k=0}^{\infty} ((b^{p^{j+1}})^{p^k} - (b^{p^{j+1}})^{p^{k+1}}) f((b^{p^{j+1}})^{p^k}) \\ &= \sum_{k=0}^{\infty} (b^{p^{k+j}} - b^{p^{k+j+1}}) f(b^{p^{k+j}}) - \sum_{k=0}^{\infty} (b^{p^{k+j+1}} - b^{p^{k+j+2}}) f(b^{p^{k+j+1}}),\end{aligned}$$

and thus,

$$\int_{b^{p^{j+1}}}^{b^{p^j}} f(x)d_px = (b^{p^j} - b^{p^{j+1}})f(b^{p^j}).$$

We now define the improper p-integral as follows.

Definition 5.1. Let $p \in (0,1)$ and $p^{-1} = b$. The improper p-integral of $f(x)$ on $[1,+\infty)$ is defined to be

$$\begin{aligned}\int_1^\infty f(x)d_px = \sum_{j=-\infty}^{\infty}\int_{b^{p^{j+1}}}^{b^{p^j}} f(x)d_px &= \sum_{j=-\infty}^{\infty}(b^{p^j} - b^{p^{j+1}})f(b^{p^j}) \\ &= \sum_{j=0}^{\infty}(b^{p^j} - b^{p^{j+1}})f(b^{p^j}) + \sum_{j=1}^{\infty}(b^{p^{-j}} - b^{p^{-j+1}})f(b^{p^{-j}}).\end{aligned}$$

Definition 5.2. If $p \in (0,1)$, then for any $j \in \{0, \pm1, \pm2, ...\}$, we have $p^{p^j} \in (0,1)$, $p^{p^j} < p^{p^{j+1}}$ and

$$\int_0^1 f(x)d_px = \sum_{j=-\infty}^{\infty}(p^{p^{j+1}} - p^{p^j})f(p^{p^j}).$$

Because, according to (4.4)

$$\int_{p^{p^j}}^{p^{p^{j+1}}} f(x)d_px = \int_0^{p^{p^{j+1}}} f(x)d_px - \int_0^{p^{p^j}} f(x)d_px = (p^{p^{j+1}} - p^{p^j})f(p^{p^j}).$$

Hence,

$$\int_0^1 f(x)d_px = \sum_{j=-\infty}^{\infty}\int_{p^{p^j}}^{p^{p^{j+1}}} f(x)d_px = \sum_{j=-\infty}^{\infty}(p^{p^{j+1}} - p^{p^j})f(p^{p^j}).$$

Definition 5.3. Let $p \in (0,1)$. Then the improper p-integral of $f(x)$ on $[0,\infty]$ is defined to be

$$\int_0^\infty f(x)d_px = \int_0^1 f(x)d_px + \int_1^\infty f(x)d_px.$$

Definition 5.4. Let $p \in (0,1)$. Then the improper p-integral of $f(x)$ on $[a,\infty]$ is defined as follows:

i) If $0 < a < 1$, then

$$\int_a^\infty f(x)d_px = \int_a^1 f(x)d_px + \int_1^\infty f(x)d_px.$$

ii) If $a > 1$, then

$$\int_a^\infty f(x)d_px = \lim_{N\to\infty}\sum_{j=1}^{N}\int_{a^{p^{-j+1}}}^{a^{p^{-j}}} f(x)d_px.$$

Here we give a sufficient condition for convergence the improper p-integral.

Proposition 5.5. Let $p \in (0,1)$, $0 < r < \infty$ and ε is a small positive number. Assume that inequality

$$\mid f(x) \mid < \min\{rx^{-\alpha}, \mid x - x^p \mid^{-1} (\ln x)^{2\alpha}\}$$

holds in neighborhood of $x = 1$ with some $0 \leq \alpha < 1$ and for sufficiently large x with some $-\varepsilon \leq \alpha < 0$. Then, the improper p-integral of $f(x)$ converges on $[1, \infty)$.

Proof. Consider $b = p^{-1}$. According to Definition 5.1, we have

$$\int_1^{\infty} f(x) d_p x = \sum_{j=0}^{\infty} (b^{p^j} - b^{p^{j+1}}) f(b^{p^j}) + \sum_{j=1}^{\infty} (b^{p^{-j}} - b^{p^{-j+1}}) f(b^{p^{-j}}).$$

By the assumptions and also Theorem 3.5, the convergence of the first sum is proved. For large x, we have $\mid f(x) \mid < \mid x - x^p \mid^{-1} (\ln x)^{2\alpha}$ where $-\varepsilon \leq \alpha < 0$. Then, we have for sufficiently large j,

$$\mid f(b^{p^{-j}}) \mid < (b^{p^{-j}} - b^{p^{-j+1}})^{-1} (\ln b^{p^{-j}})^{2\alpha}.$$

Hence

$$\begin{aligned} \mid (b^{p^{-j}} - b^{p^{-j+1}}) f(b^{p^{-j}}) \mid &\leq (b^{p^{-j}} - b^{p^{-j+1}})(b^{p^{-j}} - b^{p^{-j+1}})^{-1} (\ln b^{p^{-j}})^{2\alpha} \\ &= (\ln b^{p^{-j}})^{2\alpha} = (p^{-j} \ln b)^{2\alpha} = (\ln b)^{2\alpha} (p^{-2\alpha})^j. \end{aligned}$$

Therefore, by the comparison test, the second sum also converges. □

6 Fundamental Theorem of p-Calculus

Since we are familiar with the concepts of p-derivative and p-integral, so we're going to study the relation between them as follows. We begin this section with the following lemma.

Lemma 6.1. If $x > 1$ and $p \in (0,1)$, then $D_p I_{p^+} f(x) = f(x)$, and also if function f is continuous at $x = 1$, then we have $I_{p^+} D_p f(x) = f(x) - f(1)$.

Proof. According to definitions of p-derivative and p-integral, we have

$$I_{p^+} f(x) = \int_1^x f(s) d_p s = \sum_{j=0}^{\infty} (x^{p^j} - x^{p^{j+1}}) f(x^{p^j}).$$

Hence

$$\begin{aligned} D_p I_{p^+} f(x) &= \frac{I_{p^+} f(x^p) - I_{p^+} f(x)}{x^p - x} \\ &= \frac{\sum_{j=0}^{\infty} (x^{p^{j+1}} - x^{p^{j+2}}) f(x^{p^{j+1}}) - \sum_{j=0}^{\infty} (x^{p^j} - x^{p^{j+1}}) f(x^{p^j})}{x^p - x} \\ &= \frac{[(x^p - x^{p^2}) f(x^p) + (x^{p^2} - x^{p^3}) f(x^{p^2}) + ...] - [(x - x^p) f(x) + (x^p - x^{p^2}) f(x^p) + ...]}{x^p - x} \\ &= \frac{(x^p - x) f(x)}{x^p - x} = f(x). \end{aligned}$$

Also

$$\begin{aligned}
I_{p^+} D_p f(x) &= \lim_{N\to\infty} \sum_{j=0}^{N} (x^{p^j} - x^{p^{j+1}}) D_p f(x^{p^j}) \\
&= \lim_{N\to\infty} \sum_{j=0}^{N} (x^{p^j} - x^{p^{j+1}}) \Big(\frac{f(x^{p^{j+1}}) - f(x^{p^j})}{x^{p^{j+1}} - x^{p^j}} \Big) \\
&= \lim_{N\to\infty} \sum_{j=0}^{N} (f(x^{p^j}) - f(x^{p^{j+1}})) \\
&= \lim_{N\to\infty} (f(x) - f(x^{p^{N+1}})) = f(x) - f(1). \ \square
\end{aligned}$$

The last equality is true because f is continuous at $x = 1$. Similarly, it is easy to obtain the following lemmas.

Lemma 6.2. If $x, p \in (0, 1)$, then $D_p I_{p^-} f(x) = -f(x)$, and also if function f is continuous at $x = 1$, then we have $I_{p^-} D_p f(x) = f(1) - f(x)$.

Lemma 6.3. If $x, p \in (0, 1)$ and $I_p f(x) = \int_0^x f(s) d_p s$, then $D_p I_p f(x) = f(x)$ and also if function f is continuous at $x = 0$, then we have $I_p D_p f(x) = f(x) - f(0)$.

We are now in a position to express fundamental theorem for p-calculus.

Theorem 6.4. (Fundamental theorem of p-calculus) Let $p \in (0, 1)$. If $F(x)$ is an antiderivative of $f(x)$ and $F(x)$ is continuous at $x = 0$ and $x = 1$, then for every $0 \le a < b \le \infty$, we have

$$\int_a^b f(x) d_p x = F(b) - F(a). \tag{6.1}$$

Proof. We consider the following cases.

Case 1. Let $1 < a < b$ and a,b are finite. Since $F(x)$ is an antiderivative of $f(x)$, hence $D_p F(x) = f(x)$. By Lemma 6.1, we have

$$F(x) - F(1) = I_{p^+} f(x) = \int_1^x f(s) d_p s,$$

which implies,

$$\int_1^a f(s) d_p s = F(a) - F(1), \qquad \int_1^b f(s) d_p s = F(b) - F(1).$$

Using (4.2), thus we have

$$\int_a^b f(x) d_p x = F(b) - F(a).$$

Case 2. Let $0 \leq a < b < 1$. Since $D_pF(x) = f(x)$, by Lemma 6.3, we have

$$F(x) - F(0) = I_p f(x) = \int_0^x f(s)d_ps,$$

which implies,

$$\int_0^a f(s)d_ps = F(a) - F(0), \qquad \int_0^b f(s)d_ps = F(b) - F(0).$$

Using (4.4), thus we have

$$\int_a^b f(x)d_px = F(b) - F(a).$$

Case 3. Let $0 < a < 1 < b$ and b is finite. According to Note 4.11 and also by Lemma 6.2, we have

$$\int_a^1 f(x)d_px = I_{p^-} f(a) = I_{p^-} D_pF(a) = F(1) - F(a).$$

Similarly,

$$\int_1^b f(x)d_px = I_{p^+} f(b) = I_{p^+} D_pF(b) = F(b) - F(1).$$

Thus

$$\int_a^b f(x)d_px = F(b) - F(a).$$

For $b = +\infty$, without loss of generality, we consider $a > 1$ and by the Definition 5.4, we have

$$\begin{aligned}
\int_a^{+\infty} f(x)d_px &= \lim_{N\to\infty} \sum_{j=1}^{N} \int_{a^{p^{-j+1}}}^{a^{p^{-j}}} f(x)d_px \\
&= \lim_{N\to\infty} \sum_{j=1}^{N} (F(a^{p^{-j}}) - F(a^{p^{-j+1}})) \\
&= \lim_{N\to\infty} (F(a^{p^{-N}}) - F(a)),
\end{aligned}$$

and if $\lim_{x\to\infty} F(x)$ exists, so (6.1) is true for $b = \infty$. □

Corollary 6.5. If $f(x)$ is continuous at $x = 0$ and $x = 1$, then we have

$$\int_a^b D_pf(x)d_px = f(b) - f(a).$$

Corollary 6.6. If $f(x)$ and $g(x)$ are continuous at $x = 0$ and $x = 1$, then we have

$$\int_a^b f(x)d_pg(x) = f(b)g(b) - f(a)g(a) - \int_a^b g(x^p)d_pf(x).$$

Proof. Using the product rule (2.1), we have

$$\int_a^b D_p(fg)(x)d_px = \int_a^b (f(x)D_pg(x) + g(x^p)D_pf(x))d_px.$$

By Corollary 6.5, we have

$$f(b)g(b) - f(a)g(a) = \int_a^b f(x)D_pg(x)d_px + \int_a^b g(x^p)D_pf(x)d_px.$$

Since, $D_pg(x)d_px = d_pg(x)$, thus

$$\int_a^b f(x)d_pg(x) = f(b)g(b) - f(a)g(a) - \int_a^b g(x^p)d_pf(x). \quad \square$$

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